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New Solutions in Aesthetic Field Theory

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Abstract

We have obtained a large number of solutions to the aesthetic field equations. We discuss 19 solutions which appear to lead to bounded particle systems. One of the solutions is more complex (although only slightly) than the solution discussed in detail in Muraskin and Ring (1975). The solutions we have found have varied mathematical properties.

1. *Introduction*

In our most recent work on aesthetic field theory (Muraskin and Ring, 1975) we found a solution with considerable structure (see Figures 1-4 of Muraskin and Ring, 1975). Also, the location of the maximum (minimum) center as a function of time does not lie on a straight line. We have seen that aesthetic field theory is capable of describing a simple collision process.

In our last paper, we pointed out that the solution under discussion was not general since the restrictive condition $\Gamma^t_{tk} = \Gamma^t_{kt}$ was satisfied by the initial data, and furthermore, this relation is maintained by the field equations. This suggested to us that we should search for more solutions to the field equations. In particular, we should look for solutions that do not have unwanted restrictions such as $\Gamma^t_{tk} = \Gamma^t_{kt}$.

In this paper, we discuss the results of an extensive search for new solutions. We have employed two different methods. The first involved assigning 0.1, −0.1, or 0 for the components and then permuting these assignments. We also made some selected guesses involving the assignments ±0.2 and ±0.3. The second approach involves the making of various algebraic substitutions for the components. We then look for those that cause the integrability equations to collapse into a smaller number of equations that can be easily

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solved. This is the approach used in Muraskin and Ring (1971). For our present work, we utilized the computer making use of REDUCE.

Most of the results we will describe were obtained from the SLAC 370/168 and 360/91 computers.

2. *Solutions from Permutations of ± 0.1, 0*

We studied those solutions for which $\Gamma^{\alpha}_{\beta\gamma}$ are unchanged under a cyclic permutation of the coordinates 1, 2, 3. This is one way in which we can treat the fourth component differently from the three space coordinates. We thus have

$$\Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30}$$
$$\Gamma^1_{12} = \Gamma^2_{23} = \Gamma^3_{31}$$
etc.

Thus, $\Gamma^{\alpha}_{\beta\gamma}$ has the following structure:

$$\begin{array}{llll}
\Gamma^1_{11} = A & \Gamma^1_{12} = C_1 & \Gamma^1_{13} = C_2 & \Gamma^1_{10} = \phi \\
\Gamma^1_{21} = D_1 & \Gamma^1_{22} = E_2 & \Gamma^1_{23} = B_1 & \Gamma^1_{20} = B_4 \\
\Gamma^1_{31} = D_2 & \Gamma^1_{32} = B_2 & \Gamma^1_{33} = E_1 & \Gamma^1_{30} = B_7 \\
\Gamma^1_{01} = \theta & \Gamma^1_{02} = B_3 & \Gamma^1_{03} = B_6 & \Gamma^1_{00} = E \\
\Gamma^2_{11} = E_1 & \Gamma^2_{12} = D_2 & \Gamma^2_{13} = B_2 & \Gamma^2_{10} = B_7 \\
\Gamma^2_{21} = C_2 & \Gamma^2_{22} = A & \Gamma^2_{23} = C_1 & \Gamma^2_{20} = \phi \\
\Gamma^2_{31} = B_1 & \Gamma^2_{32} = D_1 & \Gamma^2_{33} = E_2 & \Gamma^2_{30} = B_4 \\
\Gamma^2_{01} = B_6 & \Gamma^2_{02} = \theta & \Gamma^2_{03} = B_3 & \Gamma^2_{00} = E \\
\Gamma^3_{11} = E_2 & \Gamma^3_{12} = B_1 & \Gamma^3_{13} = D_1 & \Gamma^3_{10} = B_4 \\
\Gamma^3_{21} = B_2 & \Gamma^3_{22} = E_1 & \Gamma^3_{23} = D_2 & \Gamma^3_{20} = B_7 \\
\Gamma^3_{31} = C_1 & \Gamma^3_{32} = C_2 & \Gamma^3_{33} = A & \Gamma^3_{30} = \phi \\
\Gamma^3_{01} = B_3 & \Gamma^3_{02} = B_6 & \Gamma^3_{03} = \theta & \Gamma^3_{00} = E \\
\Gamma^0_{11} = \psi & \Gamma^0_{12} = B_5 & \Gamma^0_{13} = B_8 & \Gamma^0_{10} = D \\
\Gamma^0_{21} = B_8 & \Gamma^0_{22} = \psi & \Gamma^0_{23} = B_5 & \Gamma^0_{20} = D \\
\Gamma^0_{31} = B_5 & \Gamma^0_{32} = B_8 & \Gamma^0_{33} = \psi & \Gamma^0_{30} = D \\
\Gamma^0_{01} = C & \Gamma^0_{02} = C & \Gamma^0_{03} = C & \Gamma^0_{00} = A'
\end{array} \tag{2.1}$$

Γ^i_{jk} is gotten from $\Gamma^{\alpha}_{\beta\gamma}$, using

$$\Gamma^i_{jk} = e_{\alpha}{}^i e^{\beta}{}_j e^{\gamma}{}_k \, \Gamma^{\alpha}_{\beta\gamma} \tag{2.2}$$

We chose $e^{\alpha}{}_i$ to be the following:

$$\begin{array}{llll} e^1{}_1 = 0.88 & e^1{}_2 = -0.42 & e^1{}_3 = -0.32 & e^1{}_0 = 0.22 \\ e^2{}_1 = 0.5 & e^2{}_2 = 0.7 & e^2{}_3 = -0.425 & e^2{}_0 = 0.3 \\ e^3{}_1 = 0.2 & e^3{}_2 = -0.55 & e^3{}_3 = 0.89 & e^3{}_0 = 0.6 \\ e^0{}_1 = 0.44 & e^0{}_2 = -0.16 & e^0{}_3 = 0.39 & e^0{}_0 = 1.01 \end{array} \tag{2.3}$$

The 64 $\Gamma^{\alpha}_{\beta\gamma}$ are from (2.1), given in terms of 22 parameters.

We have tested whether the integrability equations are satisfied for the following guesses:

(a) The 22 parameters can take on only the values +0.1 and −0.1. All possibilities consistent with this were considered.

(b) The 22 parameters can take on only the values to +0.1 and 0. All possibilities consistent with this were considered.

(c) The following parameters were fixed in the manner below:

$$\begin{aligned} A' &= 0.1 \\ \theta &= 0.1 \\ \phi &= 0.1 \\ \psi &= -0.1 \\ B_1 &= 0.1 \end{aligned} \tag{2.4}$$

The remaining parameters were assigned the values to +0.1, −0.1, and 0 in all possible ways. The reason we fixed θ, ϕ, ψ, A' to be nonzero is that from previous experience we noticed that all our bounded particle solutions (Muraskin, 1973; Muraskin, 1974; Muraskin and Ring, 1975) had this property. Furthermore, if we set $\psi = 0$ in Muraskin (1973), we still got a solution to the integrability equations, but we have not been able to establish a trend towards boundedness in our computer studies.

(d) We made selected trials with the following fixed elements:

B_1	B_2	B_4	B_7	A'	θ	ϕ	ψ
0.1	−0.1	−0.1	0.1	−0.1	0.1	0.1	−0.1
0.1	−0.1	−0.1	0.1	−0.1	0.1	−0.1	0.1
0.1	−0.1	−0.1	0.1	0.1	0.1	−0.1	−0.1
0.1	−0.1	−0.1	0.1	0.1	−0.1	0.1	0.1
0.1	−0.1	−0.1	0.1	0.1	0.1	−0.1	0.1
0.1	−0.1	−0.1	0.1	−0.1	0.1	0.1	0.1

B_1, B_2, B_4, B_7 are kept the same in all these instances. The signs of A', θ, ϕ, and ψ are mixed up, so as to get away from the restrictions in (2.4). The remaining parameters were then assigned 0.1, −0.1 and 0 in all possible ways.

When this was finished, we repeated the procedure for different choices of B_1, B_2, B_4, B_7 given below:

B_1	B_2	B_4	B_7
0.1	−0.1	0	0
0.1	−0.1	0.1	0.1
0.1	0	0	0
0	0	0.1	0
0	0	0.1	−0.1
0	0	0	0
−0.1	0	0	0
0	0	−0.1	0
0.1	−0.1	0	0.1
0.1	0	−0.1	0.1
0.1	0	0.1	0.1
0	0	−0.1	0.1
0.1	−0.1	0.1	−0.1
−0.1	0.1	0.1	−0.1

(e) We used the same pattern for B_1, B_2, B_4, B_7 as above in (d) but with

A'	θ	ϕ	ψ
0.1	0.1	0.2	0.1
−0.1	0.1	0.2	0.1
0.1	−0.1	0.2	0.1
0.1	0.1	0.2	−0.1
−0.1	−0.1	0.2	0.1
−0.1	0.1	0.2	−0.1
−0.1	−0.1	0.2	−0.1
0.1	−0.1	0.2	−0.1

This procedure was then repeated for $\phi = -0.2, +0.3, -0.3$. Some additional B's were tried in the ±0.2 to ±0.3 runs:

B_1	B_2	B_4	B_7
0.1	0	0.1	0
0.1	0.1	0.1	−0.1
0.1	0.1	0.1	0.1

This tremendous number of guesses has paid off and led to a large number of solutions. In Table I, we list many of the bounded particle solutions. We have not put all the bounded particle solutions we found into the table, since the solutions tend to fall into classes, the members of a class being not significantly different from other members of the same class. As an example of members of a class, we have included datum 12 in the table. This datum is similar to datum 1. The nonzero components of 12 are the same as the nonzero components of 1, only certain signs are different. Data 1 and 12 behave

TABLE 1. Solutions of integrability equations

Solution	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	A	A'	C_1	C_2	D_1	D_2	E_1	E_2	θ	ϕ	ψ	C	D	E	Type of integrability
1	0.1	−0.1	0	−0.1	−0.1	0	0.1	0.1	0.1	0.1	0.1	0	0	0.1	0	0	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} \neq 0$
2	0.1	−0.1	0	−0.1	−0.1	0	0.1	−0.1	0.1	0.1	0	0.1	0.1	0	0	0	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} = 0$
3	0	0	0	0	−0.1	0	0	−0.1	0.1	0.1	0	0	0.1	0.1	0	0	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} \neq 0$
4	0.1	−0.1	0	0	−0.1	0	0	−0.1	0	0.1	−0.1	0	0.1	0	0	0	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} \neq 0$
5	0.1	0	0	−0.1	−0.1	0	0.1	−0.1	0	0.1	−0.1	0	0.1	0.1	0.1	0	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} = 0$
6	0.1	0	0.1	0.1	−0.1	0.1	0.1	−0.1	0	0.1	0	0	0.1	0	0	0.1	0.1	0.1	−0.1	−0.1	−0.1	0.1	$R^i_{jkl} = 0$
7	0	0	0	0.1	−0.1	0	−0.1	−0.1	0.1	0.1	0.1	0	0.1	0	−0.1	0.1	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} = 0$
8	0	0	0	0.1	0	0	−0.1	0	0	0.1	0	0	−0.1	0.1	0.1	−0.1	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} \neq 0$
9	0	0	0	0	0	0	0	0.1	−0.1	0.1	−0.1	0	0	−0.1	0	0	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} \neq 0$
10	0.1	−0.1	0	0	0	0	0	0	0	0.1	0	0	0	0	0	0	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} = 0$
11	0.1	−0.1	0	0.1	−0.1	0	0.1	−0.1	0.1	0.1	0	0.1	0.1	0	0	0	0.1	−0.1	−0.1	0	0	0	$R^i_{jkl} = 0$
12	0.1	−0.1	0	0.1	0.1	0	−0.1	−0.1	−0.1	0.1	0	−0.1	−0.1	0	0	0	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} \neq 0$
13	−0.1	0.1	0.1	0.1	−0.1	0.1	0.1	−0.1	−0.1	0.1	−0.1	−0.1	−0.1	0.1	0.1	−0.1	0.1	0.1	−0.1	0.1	0.1	−0.1	$R^i_{jkl} = 0$
14	−0.1	0.1	0.1	0.1	−0.1	0.1	0.1	−0.1	0.1	0.1	0.1	0.1	−0.1	0.1	0.1	−0.1	0.1	0.1	−0.1	−0.1	−0.1	0.1	$R^i_{jkl} = 0$
15	−0.1	0.1	0	0.1	0	0	−0.1	0	0.1	0	0	0	0	0	0	0	0.1	0.1	−0.1	0	0	0	$R^i_{jkl} = 0$
16	0	0	0	0	0.1	0	0.1	0.1	0.1	−0.1	0.1	0	0.1	0	−0.1	0.1	−0.1	−0.2	0.1	0	0	0	$R^i_{jkl} = 0$
17	0.1	−0.1	0	0.1	0.1	0	0.1	0.1	0.1	−0.1	0	0.1	0.1	0	0	0	−0.1	−0.3	0.1	0	0	0	$R^i_{jkl} = 0$
18	0	0	0.1	0	−0.1	0.1	0.1	−0.1	0.1	0.1	0.1	0.1	0	0	0	0	0.1	0.2	−0.1	−0.1	−0.1	0.1	$R^i_{jkl} = 0$
19	0	0	0.1	0.1	−0.1	0.1	−0.1	−0.1	0.1	0.1	0.1	0.1	0	0	0	0	0.1	0.3	−0.1	−0.1	−0.1	0.1	$R^i_{jkl} = 0$

similarly in computer runs. Another way that a datum can get to be in the same class is if it has just about all the components the same except for a few that are zero in one case and not zero in another.[1] Thus, some judgement was made in order to not make the table too cumbersome.

We see that there are many different bounded particle solutions to the field equations with diverse properties for $\Gamma^{\alpha}_{\beta\gamma}$. We shall have more to say about the solutions after we discuss our second approach to finding solutions.

3. *Algebraic Approach to Finding Solutions*

We still make use of the structure (2.1). We perform the algebraic substitutions

$$\begin{aligned} &B_3 = B_6 = \theta && B_8 = B_5 = \psi \\ &B_4 = B_7 = \phi && B_1 = E_2 = D_1 \\ &A = C_1 = C_2 && \phi = \theta \\ &B_2 = E_1 = D_2 && \end{aligned} \tag{3.5}$$

The $R^{i}{}_{jkl} = 0$ integrability equations then collapse into the following smaller set:

$$\begin{aligned} &C\phi - 2D\phi + E\psi = 0 \\ &2CE - DE - ED_1 - EC_1 - ED_2 - A'\phi + 3\phi^2 = 0 \\ &D^2 - DD_1 - DC_1 - DD_2 - A'\psi + 3\psi\phi = 0 \\ &CA' - DA' + 3\phi D - 3E\psi = 0 \end{aligned} \tag{3.2}$$

The solution of these equations is

$$\begin{aligned} &C = D \\ &\psi = \phi D/E \\ &A' = \frac{3\phi^2 + E(D - C_1 - D_1 - D_2)}{\phi} \end{aligned} \tag{3.3}$$

We choose as parameters, the following:

$$\begin{aligned} &C_1 = -0.2, && D = 5/4 \\ &D_1 = -0.1, && \phi = 1 \\ &D_2 = -0.05, && E = -5/4 \end{aligned} \tag{3.4}$$

Then, we get $C = 5/2$, $\psi = 1$, $A' = 1$. The solution has $\Gamma^{t}_{tk} \neq \Gamma^{t}_{kt}$ and leads to a bounded particle.

[1] Maps for some solutions look similar to other solutions found. This, too, was used to shorten the table.

An algebraic substitution that does not lead to $\theta = \phi$ is as follows:

$$\begin{aligned} &\theta = B_3 = B_6, \qquad && D_1 = B_1 = E_2 \\ &\psi = B_5 = B_8, && D_2 = B_2 = E_1 \\ &A = C_1 = C_2, && B_4 = -B_7 \end{aligned} \tag{3.5}$$

In this case, the $R^i{}_{jkl} = 0$ integrability equations again collapse and can be solved. We find

$$\begin{aligned} &D = C \qquad && A' = \frac{3\theta^2 + CE - E(C_1 + D_1 + D_2)}{\theta} \\ &\psi = D\theta/E && \\ &\phi = 3\theta && \end{aligned} \tag{3.6}$$

We choose

$$\begin{aligned} &C_1 = 0.09 \qquad && \phi = 0.6 \\ &D_1 = -0.04 && D = 0.1 \\ &D_2 = -0.05 && E = -0.1 \end{aligned} \tag{3.7}$$

This leads to $\theta = 0.2$, $C = 0.1$, $\psi = -0.2$, $A' = 0.55$. This set of data also has $\Gamma^t_{tk} \neq \Gamma^t_{kt}$ and leads to a bounded particle.

4. *Results*

We can draw the following results from our investigations. Note that datum 1 was discussed in greater detail in Muraskin and Ring (1975). Datum 10 appears in Muraskin (1973), datum 15 in Muraskin (1974), and datum 3 in Muraskin and Ring (1976).

(a) There are many solutions to the integrability equations.

(b) There are many different ways to obtain a bounded particle system with all sorts of different properties: $R^i{}_{jkl} = 0$ or not; $\Gamma^t_{tk} = \Gamma^t_{kt}$ or not; $g_{\alpha\beta}\Gamma^\alpha_{\gamma\delta}$ having (not having) a totally antisymmetric part when $g_{\alpha\beta} = (1, 1, 1, 1)$. Still others can have (not have) a maximum in g_{00} at the origin when $g_{\alpha\beta} = (1, 1, 1, 1)$.

(c) However, even though the solutions for the table have diverse properties for $\Gamma^\alpha_{\beta\gamma}$, the characteristics of the solutions have much in common with one another. Our results are inferred from maps around the origin and long runs down the $\pm x$ axis. (For datum 9, runs were made along the $\pm y$ axis instead of $\pm x$). The field goes to zero if one gets far enough down the axis. The number of turnabout points down an axis is an important criterion of how complex a solution is. That is, a many-body system would be expected to have a large number of turnabout points down an axis. In no case did we find more than 4 turnabout points down an axis, and 4 was not common.

Another criterion for complexity is the number of planar maxima and minima. Previously, datum 1 was the most complex we had found with 4 such planar maxima and minima (see Figure 1 of Muraskin and Ring, 1975). We have found a solution with greater complexity. Datum 16 has 5 such planar maxima and minima. This increase in complexity is not overly dramatic.

We have not had the computer capability to study the trajectories of the particles for the many solutions.

From our work this far, we see it is not so simple a task to increase the complexity of solutions.

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Note added in proof

See also M. Muraskin (1975). *International Journal of Theoretical Physics*, **13**, 303. In this paper equation 3.13 should read

$$\Gamma^i_{jk} = e_\alpha{}^i \frac{\partial e^\alpha{}_j}{\partial x^k}$$

and 3.14 should read

$$de_\alpha{}^i = -\Gamma^i_{jk} e_\alpha{}^j dx^k$$